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Nonlinear dynamics of mushy layers induced by external stochastic fluctuations

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The time-dependent process of directional crystallization in the presence of a mushy layer is considered with allowance for arbitrary fluctuations in the atmospheric temperature and friction velocity. A nonlinear set of mushy layer equations and boundary conditions is solved analytically when the heat and mass fluxes at the boundary between the mushy layer and liquid phase are induced by turbulent motion in the liquid and, as a result, have the corresponding convective form. Namely, the 'solid phase–mushy layer' and 'mushy layer–liquid phase' phase transition boundaries as well as the solid fraction, temperature and concentration (salinity) distributions are found. If the atmospheric temperature and friction velocity are constant, the analytical solution takes a parametric form. In the more common case when they represent arbitrary functions of time, the analytical solution is given by means of the standard Cauchy problem. The deterministic and stochastic behaviour of the phase transition process is analysed on the basis of the obtained analytical solutions. In the case of stochastic fluctuations in the atmospheric temperature and friction velocity, the phase transition interfaces (mushy layer boundaries) move faster than in the deterministic case. A cumulative effect of these noise contributions is revealed as well. In other words, when the atmospheric temperature and friction velocity fluctuate simultaneously due to the influence of different external processes and phenomena, the phase transition boundaries move even faster.

This article is part of the theme issue 'From atomistic interfaces to dendritic patterns'.

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1. Introduction

Crystallization phenomena occur frequently in many geophysical and metallurgical processes and completely determine their dynamics and the physical properties of solidified materials [1–7]. A number of such processes can happen in the presence of a matrix, or layer of mixed state, that is filled with the solid and liquid phases. A theoretical description of this layer, termed a 'mush' or 'mushy layer', was given a while back by Hills *et al.* [8]. They formulated a nonlinear set of mushy layer equations and solved a much-reduced set of them approximately for the constrained crystallization regime of a binary system. Shortly thereafter Fowler found a more complete approximate solution of these equations [9]. The self-similar solidification scenario has since been described by Worster [10] and Alexandrov and co-workers [11–13] for binary and ternary systems. Different analytical solutions for the steady-state crystallization conditions with a mush have also been obtained in a series of papers [14–18]. Some approximate analytical solutions [19–22] describing the time-dependent mushy layer evolution processes for various crystallization conditions should be mentioned as well. An important point in all of these studies lies in the fact that the heat and mass fluxes at the phase transition boundaries were given in a standard (conductive) form. However, there exist processes where the heat and mass fluxes should be chosen in convective form. Such situations take place, for instance, when we are dealing with the evolution of sea ice in the presence of substantial oceanic flows [23–25]. The present paper is concerned with complete analytical solutions of the mushy layer equations that describe this crystallization scenario. In addition, we analyse below how possible stochastic fluctuations alter the deterministic behaviour of the obtained solutions.

2. Theoretical background

Let us consider the process of directional crystallization of a binary mixture (e.g. ice and water or metallic melt) along the spatial direction z (figure 1). The total phase transition domain is divided into three parts: solid phase $0 < z < a(t)$, mushy layer $a(t) < z < b(t)$ and liquid phase $z > b(t)$. Here t is the crystallization time and $a(t)$ and $b(t)$ represent the 'solid phase–mushy layer' and 'mushy layer–liquid phase' phase transition boundaries, respectively. As this takes place, the mushy layer is filled with the solid and liquid material. In addition, solid phase structures in the form of dendrites, nuclei and more complex formations evolve in such a layer and completely compensate its supercooling. In this case, the mushy layer is described by means of the quasi-equilibrium model [8–10]. We consider the case when the temperature distributions in the solid phase (T_s) and mushy layer (T_m) can be approximated by the linear functions

$$\left.\begin{aligned} T_s(z,t) &= T_0(t) + C_1(t)z, \quad 0 < z < a(t), \\ \text{and} \qquad T_m(z,t) &= T_1(t) + T_2(t)z, \quad a(t) < z < b(t), \end{aligned}\right\} \tag{2.1}$$

where $T_0(t)$ is the temperature at $z=0$ (e.g. atmospheric temperature), and $C_1(t)$, $T_1(t)$ and $T_2(t)$ represent the time-dependent functions found below. The linear form of the temperature profiles (2.1) follows from the fact that the temperature relaxation time is many times smaller than a characteristic time of the phase interface motion or than a characteristic relaxation time of solute diffusion. Note that a linear temperature distribution has been observed in a number of experiments (see, among others, [26–29]).

Let us describe the mass transport in the mushy layer by means of the Scheil equation [30,31]:

$$\frac{\partial}{\partial t}((1-\varphi)C_m) = 0, \quad a(t) < z < b(t), \tag{2.2}$$

where $\varphi(z,t)$ is the solid fraction and $C_m(z,t)$ stands for the solute concentration (brine salinity in the case of sea water and ice). Note that equation (2.2) satisfies experimental data in a broad range of experimental conditions [1].

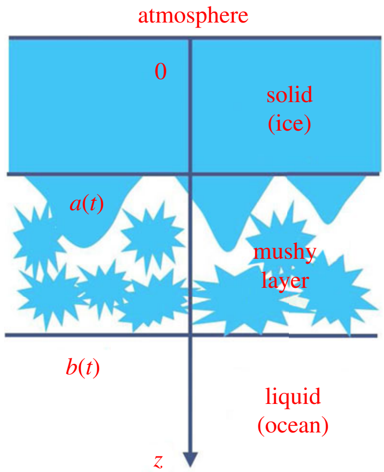

Figure 1. A schematic of the crystallization process. (Online version in colour.)

The temperature and solute concentration in the mushy layer are connected through the liquidus line equation

$$T_\mathrm{m} = T_* - mC_\mathrm{m}, \quad a(t) < z < b(t), \tag{2.3}$$

where T_* is the phase transition temperature at $C_\mathrm{m} = 0$ and m is the equilibrium liquidus slope.

At the solid phase–mushy layer interface, we have the following boundary conditions [10,32]:

$$L_V(1-\varphi_a)\frac{\mathrm{d}a}{\mathrm{d}t} = k_\mathrm{s}\frac{\partial T_\mathrm{s}}{\partial z} - k_\mathrm{m}(\varphi_a)\frac{\partial T_\mathrm{m}}{\partial z}, \quad z = a(t), \tag{2.4}$$

$$C_\mathrm{m}(1-\varphi_a)\frac{\mathrm{d}a}{\mathrm{d}t} = -D_\mathrm{m}(\varphi_a)\frac{\partial C_\mathrm{m}}{\partial z}, \quad z = a(t), \tag{2.5}$$

and

$$\varphi = \varphi_a, \quad T_\mathrm{s} = T_\mathrm{m}, \quad z = a(t), \tag{2.6}$$

where L_V is the latent heat of phase transition, $\varphi_a(t)$ is the solid fraction at $z = a(t)$, $k_\mathrm{m}(\varphi_a) = k_\mathrm{s}\varphi_a + k_\mathrm{l}(1-\varphi_a)$, k_s and k_l are the thermal conductivities of the solid and liquid phases, $D_\mathrm{m}(\varphi_a) = D_\mathrm{l}(1-\varphi_a)$, and D_l is the solute diffusivity. Here, we assume that the thermal conductivity of the mushy layer, $k_\mathrm{m}(\varphi) = k_\mathrm{s}\varphi + k_\mathrm{l}(1-\varphi)$, is the volume-fraction-weighted average of the properties of the liquid and solid phases [10,33–35].

In the case of heavy oceanic currents, the boundary conditions at the mushy layer–liquid phase interface must be written down accounting for the fact that the heat and salt fluxes in the liquid phase (ocean) are strongly dependent on turbulence (convective mixing in liquid) [25,36,37]. Taking this into consideration, we get

$$L_V\varphi_b\frac{\mathrm{d}b}{\mathrm{d}t} = k_\mathrm{m}(\varphi_b)\frac{\partial T_\mathrm{m}}{\partial z} - \alpha_\mathrm{h}\rho_\mathrm{l}c_\mathrm{l}u(T_\infty - T_b), \quad z = b(t), \tag{2.7}$$

$$C_\mathrm{m}\varphi_b\frac{\mathrm{d}b}{\mathrm{d}t} = D_\mathrm{l}(1-\varphi_b)\frac{\partial C_\mathrm{m}}{\partial z} - \alpha_s u(C_\infty - C_\mathrm{m}), \quad z = b(t), \tag{2.8}$$

and

$$\varphi = \varphi_b, \quad T_\mathrm{m} = T_b, \quad z = b(t), \tag{2.9}$$

where α_h and α_s are the turbulent transfer coefficients for heat and salt, ρ_l and c_l are the density and specific heat of the liquid phase, u is the friction velocity (in the case of time-dependent oceanic currents, it is a function of time t) and T_∞ and C_∞ are the temperature and solute concentration (salinity) in liquid far from the moving boundary $b(t)$. Note that the ratio of transfer coefficients $\alpha_\mathrm{h}/\alpha_\mathrm{s} = (a_\mathrm{l}/D_\mathrm{l})^n$ with $\frac{2}{3} < n < \frac{4}{5}$ [27,38,39], where a_l is the thermal diffusivity in liquid.

3. Analytical solutions

In this section, we construct analytical solutions of the heat and mass transfer equations (2.1)–(2.9) describing the nonlinear dynamics of a mushy layer.

Integration of equation (2.2) with allowance for expressions (2.1), (2.3) and (2.9) leads to the following distribution of the solid phase:

$$\varphi(z,t) = 1 + \frac{(\varphi_b - 1)(T_b - T_*)}{T_1(t) + zT_2(t) - T_*}. \tag{3.1}$$

Next substituting distributions (2.1), (2.3) and (3.1) into the boundary conditions (2.4)–(2.9), we obtain

$$\varphi_a(t) = 1 + \frac{(\varphi_b - 1)(T_b - T_*)}{T_0(t) + a(t)C_1(t) - T_*}, \tag{3.2}$$

$$C_1(t) = \frac{L_V(1-\varphi_a)}{k_s}\frac{\mathrm{d}a}{\mathrm{d}t} + \frac{k_m(\varphi_a)T_2(t)}{k_s}, \tag{3.3}$$

$$T_0(t) + a(t)C_1(t) = T_b + (a-b)T_2(t), \tag{3.4}$$

$$[T_* - T_b + (b-a)T_2(t)]\frac{\mathrm{d}a}{\mathrm{d}t} = D_l T_2(t), \tag{3.5}$$

$$T_1(t) = T_b - b(t)T_2(t), \tag{3.6}$$

$$k_l T_2(t) = \alpha_h \rho_l c_l u(T_\infty - T_b) \tag{3.7}$$

and

$$-D_l T_2 = \alpha_s u(mC_\infty - T_* + T_b). \tag{3.8}$$

Here, we assume that $\varphi_a \neq 1$ and $\varphi_b = 0$.

Now combining equations (3.2) and (3.4), we rewrite the solid fraction φ_a in the form

$$\varphi_a(t) = 1 - \frac{T_b - T_*}{T_b - T_* + (a-b)T_2}. \tag{3.9}$$

Equations (3.7) and (3.8) determine the temperature gradient T_2 and boundary temperature T_b as

$$T_2 = \frac{\alpha_h\rho_l c_l u(T_\infty - T_b)}{k_l}, \quad T_b = \frac{T_* - mC_\infty - \kappa T_\infty}{1-\kappa}, \quad \kappa = \frac{\alpha_h \rho_l c_l D_l}{\alpha_s k_l}. \tag{3.10}$$

Eliminating now $a - b$ from equations (3.5) and (3.9), we arrive at the solid fraction

$$\varphi_a(t) = 1 + \frac{(T_b - T_*)}{D_l T_2}\frac{\mathrm{d}a}{\mathrm{d}t}. \tag{3.11}$$

The next step is to substitute (3.3) and (3.11) into (3.4) and transform the result into a cubic equation for the phase transition rate $\mathrm{d}a/\mathrm{d}t = a'$:

$$Aa'^3 + Ba'^2 + Ca' + D_l T_2 = 0, \tag{3.12}$$

where

$$A = -\frac{(T_b - T_*)L_V a}{D_l k_s T_2}, \quad B = \frac{(T_b - T_*)(1-K)a}{D_l}, \quad C = aT_2 + T_0 - T_*, \quad K = \frac{k_l}{k_s}.$$

Note that A, B and C are functions of a only in the case of crystallization without time-dependent fluctuations in the friction velocity u and when the atmospheric temperature T_0 is constant. If this is really the case, the solution of equation (3.12) can be found by Cardano's formula, i.e.

$$a' \equiv f(a) \quad \text{and} \quad t(a) = \int_{a_0}^{a} \frac{\mathrm{d}a_1}{f(a_1)}, \tag{3.13}$$

where $a_0 = a(0)$ is the initial coordinate of the phase interface. Then combining expressions (3.5), (3.11) and (3.13), we arrive at

$$\varphi_a(a) = 1 + \frac{(T_b - T_*)}{D_l T_2} f(a), \quad b(a) = a + \frac{D_l}{f(a)} + \frac{T_b - T_*}{T_2}. \tag{3.14}$$

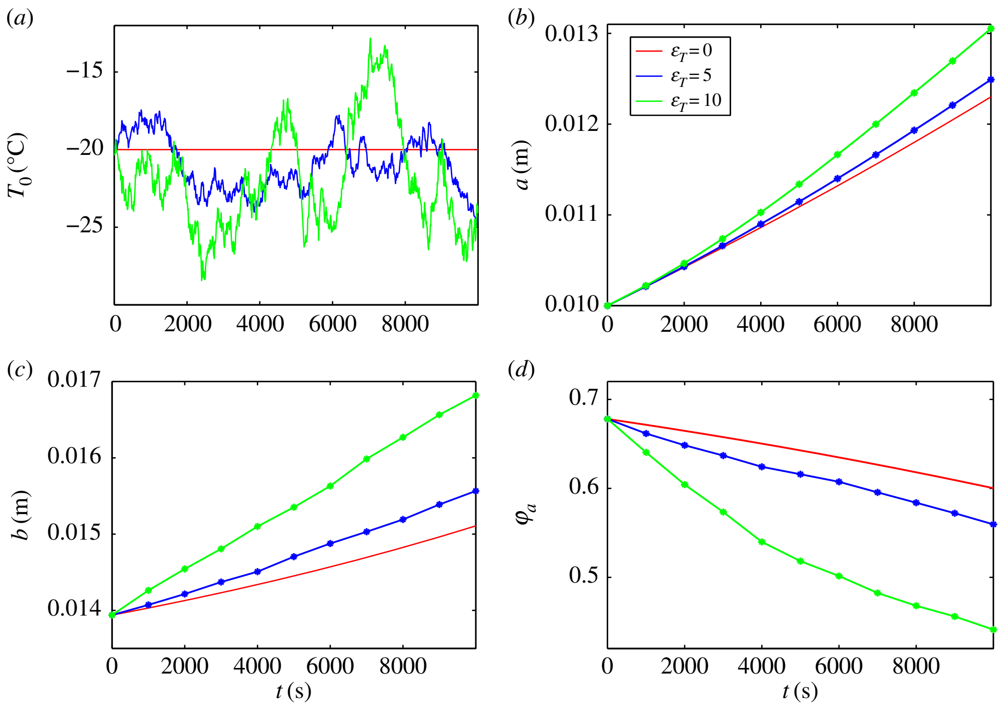

Figure 2. The influence of atmospheric temperature fluctuations (additive noise) on nonlinear crystallization dynamics: atmospheric temperature at different noise intensities (*a*), solid phase–mushy layer boundary (*b*), mushy layer–liquid phase boundary (*c*) and solid fraction at the solid phase–mushy layer boundary (*d*). Physical parameters of the sea ice–water system are [20,21,25,34]: $D_l = 1.2 \times 10^{-9}$ m^2 s^{-1}, $L_V = 3 \times 10^8$ W s m^{-3}, $k_l = 0.56$ W m^{-1} °C^{-1}, $k_s = 2.03$ W m^{-1} °C^{-1}, $T_* = 0$, $T_b = -2$°C, $\rho_l = 10^3$ kg m^{-3}, $c_l = 4187$ J kg^{-1} °C^{-1}, $C_\infty = 28$ psu, $T_\infty = 5$°C, $m = 0.052$°C psu^{-1}, $\alpha_h = 0.0095$, $\alpha_s = \alpha_h/35$, $\bar{T}_0 = -20$°C, $u = \bar{u} = 0.005$ m s^{-1}, $\varepsilon_u = 0$. (Online version in colour.)

Thus, expressions (3.13) and (3.14) determine a complete analytical solution of the problem under consideration in a parametric form (with parameter a). Note that C_1, T_1 and T_2 are defined by expressions (3.3), (3.6) and (3.10).

In a more general case when $T_0 = T_0(t)$ and $u = u(t)$, equation (3.12) leads to the following solution:

$$a' \equiv f(a, t), \quad a(0) = a_0. \tag{3.15}$$

This expression represents the standard Cauchy problem, which completely determines the boundary position $a = a(t)$. In this case, φ_a and b have the forms

$$\varphi_a(t) = 1 + \frac{(T_b - T_*)}{D_l T_2} f(a(t), t), \quad b(t) = a(t) + \frac{D_l}{f(a(t), t)} + \frac{T_b - T_*}{T_2}. \tag{3.16}$$

What is more, C_1, T_1 and T_2, as before, are given by expressions (3.3), (3.6) and (3.10). As a result, expressions (3.15) and (3.16) also represent the exact analytical solutions of the phase transition problem with a mushy layer in the presence of arbitrary time variations in $T_0 = T_0(t)$ and $u = u(t)$. An important point is that this solution remains valid in the case of stochastic variations occurring as a result of random fluctuations in the ocean and in the atmosphere. The deterministic and stochastic dynamics of a mushy layer governed by exact solutions (3.13)–(3.16) is analysed below.

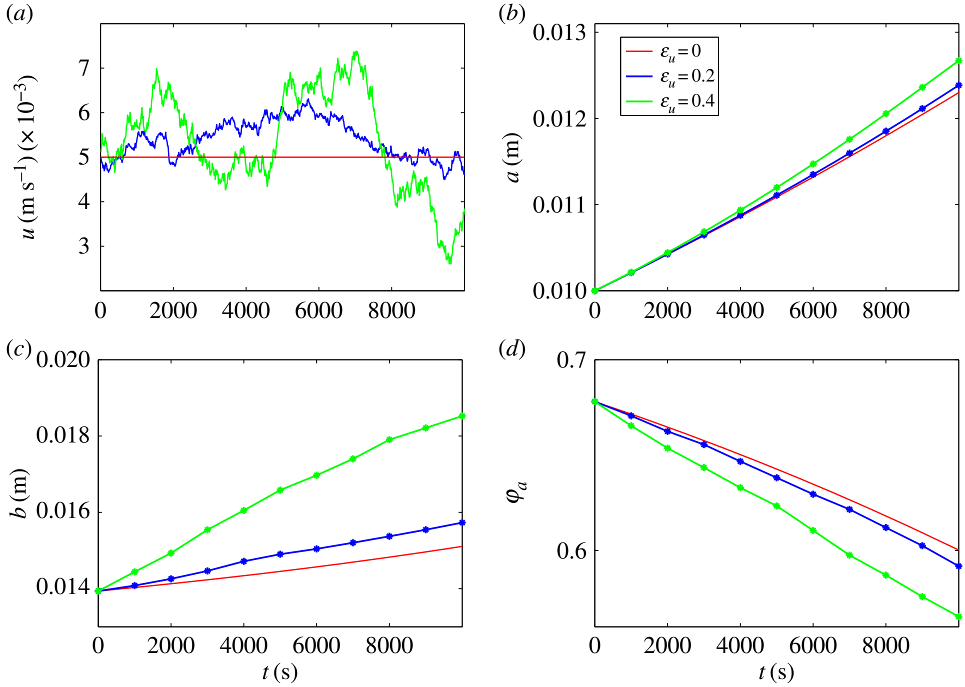

Figure 3. The influence of friction velocity fluctuations (multiplicative noise) on nonlinear crystallization dynamics: friction velocity at different noise intensities (*a*), solid phase–mushy layer boundary (*b*), mushy layer–liquid phase boundary (*c*) and solid fraction at the solid phase–mushy layer boundary (*d*). The sea ice–water parameters correspond to figure 2, and $T_0 = \bar{T}_0 = -20^\circ\mathrm{C}$, $\bar{u} = 0.005\ \mathrm{m\,s^{-1}}$, $\varepsilon_T = 0$. (Online version in colour.)

4. Deterministic and stochastic dynamics

As the surface temperature T_0 at $z = 0$ and friction velocity u in the oceanic boundary layer adjacent to the mushy layer phase interface $b(t)$ represent random functions due to completely unpredictable wind gusts and oceanic flows, let us study their influence on the sea ice dynamics on the basis of stochastic processes. First of all, let us introduce the additive and multiplicative random fluctuations in the atmospheric temperature $T_0(t) = \bar{T}_0 + \varepsilon_T \eta_T(t)$ and/or in the friction velocity $u(t) = \bar{u}(1 + \varepsilon_u \eta_u(t))$ by means of the independent Ornstein–Uhlenbeck processes $\eta_T(t)$ and $\eta_u(t)$ which are generated by the following Langevin equation [40]:

$$\dot{\eta} = -p\eta + \sqrt{2p}\,\xi(t), \tag{4.1}$$

where $\xi(t)$ is the standard white Gaussian noise, ε_T and ε_u are the intensities of stochastic fluctuations and $\bar{T}_0$ and $\bar{u}$ represent the corresponding mean values. Let us emphasize that the steady-state solution of equation (4.1) is described by the mean value function $E(\eta(t)) = 0$, stochastic variance $E(\eta(t))^2 = 1$ as well as the autocovariance function $\mathrm{cov}(\eta(t), \eta(t+\tau)) = \exp(-p\tau)$, where p is responsible for different covariance structures. In addition, if p is small enough, we get slow random fluctuations of the atmospheric temperature $T_0(t)$ and/or friction velocity $u(t)$. The frequencies of their fluctuations grow with increasing p, and if $p \to \infty$ we arrive at the white noise. In this paper, we use $p = 0.0001$.

A deterministic and stochastic dynamics in accordance with the obtained analytical solutions (3.13)–(3.16) is shown in figures 2–4. The deterministic behaviour is demonstrated by the solid lines in the absence of noise intensities, $\varepsilon_T = \varepsilon_u = 0$. The stochastic evolutionary scenarios plotted in figures 2 and 3 for different fluctuations (noise) in the atmospheric temperature $T_0(t) = \bar{T}_0 + \varepsilon_T \eta_T(t)$ (figure 2, additive noise) and friction velocity $u(t) = \bar{u}(1 + \varepsilon_u \eta_u(t))$ (figure 3, multiplicative noise) demonstrates that the phase transition boundaries $a(t)$ and $b(t)$ are very

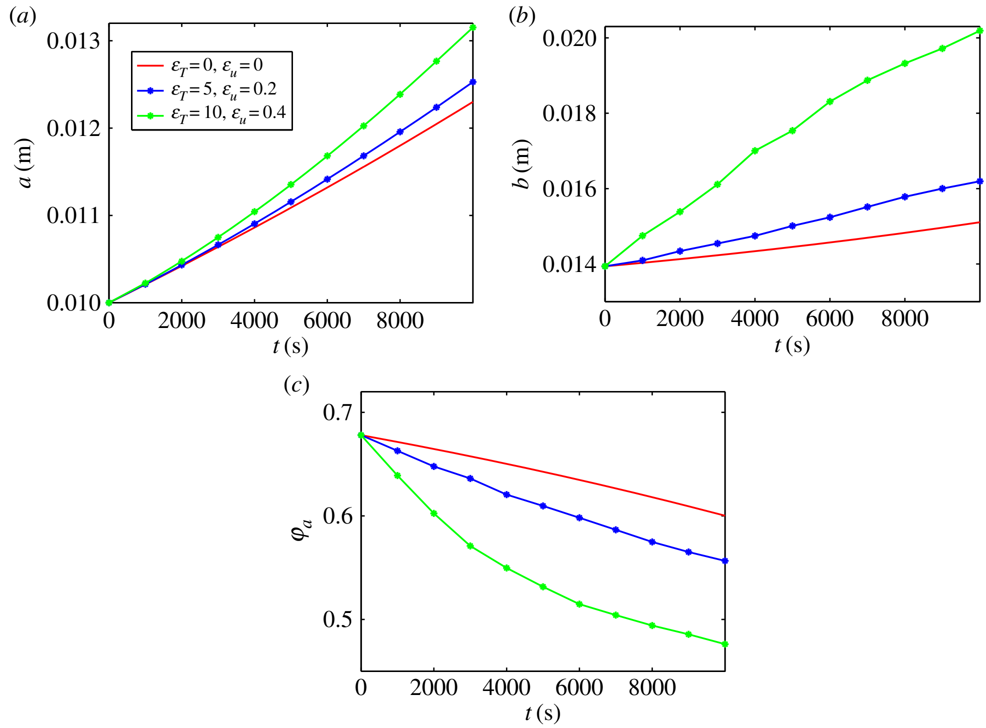

Figure 4. The cumulative influence of atmospheric temperature and friction velocity fluctuations (additive and multiplicative noise contributions) on nonlinear crystallization dynamics: solid phase–mushy layer boundary (*a*), mushy layer–liquid phase boundary (*b*) and solid fraction at the solid phase–mushy layer boundary (*c*). The sea ice–water parameters correspond to figure 2, and $\bar{T}_0 = -20^\circ$C, $\bar{u} = 0.005$ m s^{-1}. (Online version in colour.)

sensitive to random fluctuations in T_0 and u induced by external physical processes. Namely, the pure ice $a(t)$ and mushy layer $b(t)$ boundaries move more rapidly with increasing noise intensities ε_T and ε_u (the corresponding curves have been plotted as a result of time averaging over 1000 numerical experiments). This behaviour of the phase transition boundary $b(t)$ coincides with the previously studied model of mushy layer evolution where the classical (conductive) Stefan-type boundary conditions have been used [41]. In addition, both noise sources give a cumulative effect (figure 4). In other words, the phase transition boundaries propagate even faster when $T_0(t)$ and $u(t)$ fluctuate simultaneously. As this takes place, the solid fraction $\varphi_a(t)$ decreases with increasing ε_T and ε_u. This is due to the fact that the mushy layer thickness $b(t) - a(t)$ also expands with growing noise, which in turn decreases its solid fraction.

5. Conclusion

In the present paper, we constructed two exact analytical solutions of nonlinear unsteady-state mushy layer equations that describe the directional crystallization of binary mixtures (sea ice–water systems) with allowance for turbulent motion in the liquid. The first exact analytical solution that describes the freezing scenario with a constant atmospheric temperature and friction velocity is found in a parametric form. The second analytical solution corresponding to a more common freezing regime when the temperature and velocity represent arbitrary functions of time is written out in full in the form of a standard Cauchy problem. The nonlinear behaviour of the obtained solutions is analysed in cases of deterministic dynamics and stochastic fluctuations in the atmospheric temperature and friction velocity induced by external random forcing such as wind gusts and oceanic flows. We demonstrate that stochastic forcing drastically changes the mushy layer dynamics. Namely, the phase transition boundaries propagate into the liquid phase faster and faster with increasing noise intensities. What is more, if both noise sources introduced in the atmospheric temperature and friction velocity occur simultaneously, the phase interfaces move even faster. The obtained solutions extend the previously known theory [20,34,41] (which characterizes the conductive heat and mass transfer fluxes at the mushy layer–liquid phase boundary) by introducing the corresponding convective fluxes to describe the freezing processes with a mush accounting for turbulent flows in the oceanic boundary layer.

Data accessibility. This article has no additional data.

Authors' contributions. All authors contributed equally to the present research article.

Competing interests. We declare we have no competing interests.

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